TRANSONIC FLOW OF A VISCOUS GAS IN A PLANE-PARALLEL CHANNEL

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Results are given of a theoretical analysis of an adiabatic gas flow in a plane-parallel channel near the critical point. It is shown that the velocity profile ahead of the critical point must fill out, and that the mean velocity at the critical section exceeds the velocity of sound.

Consider a steady adiabatic subsonic gas flow at Pr = 1 in a plane-parallel channel. We use the following reference values: length, h; limiting flow velocity, w_l; stagnation temperature; stagnation density and pressure at the channel inlet, ρ_{0i}, p_{0i}; the gas constant R for the entropy per unit mass and the value $2h\rho_{0i}w_l$ for mass flow. Then all the quantities will be dimensionless.

We denote the longitudinal and transverse velocity components by u and v, respectively, and put $V^2 = u^2 + v^2$.

As the stagnation temperature is constant, we have

$$T = 1 - V^2. \tag{1}$$

The pressure, density, and temperature of the gas are related by the equation of state

$$p = \rho T. \tag{2}$$

Let us examine the variation of entropy s along a certain stream-line. It is convenient to use a curvilinear system of coordinates, along the stream-lines and at right angles to them.

The position of an arbitrary point M is determined by the length σ, calculated from the inlet section of the channel along the stream-line passing through M, and the length Σ at right angles, calculated from one wall of the channel.

Taking the heat capacity of the gas to be constant, from the first law of thermodynamics we have

$$\frac{\partial s}{\partial \sigma} = -\frac{1}{p}\frac{\partial p}{\partial \sigma} + \frac{k}{k-1}\frac{1}{T}\frac{\partial T}{\partial \sigma}, \tag{3}$$

or, taking into account (2) and (1)

$$\frac{\partial s}{\partial \sigma} = -\frac{1}{\rho}\frac{\partial \rho}{\partial \sigma} - \frac{2}{k-1}\frac{V}{1-V^2}\frac{\partial V}{\partial \sigma}. \tag{4}$$

In this coordinate system the equation of continuity takes the form

$$\frac{1}{\rho}\frac{\partial \rho}{\partial \sigma} + \frac{1}{V}\frac{\partial V}{\partial \sigma} + \frac{\partial \varphi}{\partial \Sigma} = 0, \tag{5}$$

where $\mathrm{tg}\varphi = u/v$.

If $\partial\varphi/\partial\Sigma > 0$ at a certain point, the stream-lines diverge there, and if $\partial\varphi/\partial\Sigma < 0$ the stream-lines converge.

Using (5), we may put (4) in the form

$$\frac{\partial s}{\partial \sigma} = \frac{1-(k+1)V^2/(k-1)}{V(1-V^2)}\frac{\partial V}{\partial \sigma} + \frac{\partial \varphi}{\partial \Sigma}. \tag{6}$$

From the second law of thermodynamics $\partial s/\partial\sigma \geq 0$, since we are considering a thermally insulated flow.

It follows from (6) that the subsonic velocity $V^2 < (k-1)/(k+1)$ increases when $\partial\varphi/\partial\Sigma < \partial s/\partial\sigma$. If the velocity V exceeds the velocity of sound, then it may increase further only if $\partial\varphi/\partial\Sigma > \partial s/\partial\sigma \geq 0$, i.e., if the stream-lines diverge.

As the critical speed is approached, the one-dimensional theory and also experiment [1, 2] indicate that the mean velocity increases, approaching the speed of sound. Since the velocity is zero at the wall, as the critical point is approached, the velocity near the channel axis must increase and be greater than the speed of sound. Therefore near the channel axis the stream-lines must diverge.

Taking limiting conditions sufficiently accurate for this case

$$\left|\frac{\partial p}{\partial x}\right| \gg \left|\frac{\partial p}{\partial y}\right|, \quad u \gg |v|, \tag{7}$$

we shall show that the velocity profile fills out as the critical point is approached.

Bearing in mind that because of (7)

$$\frac{\partial}{\partial \sigma} = \frac{u}{V}\frac{\partial}{\partial x} + \frac{v}{V}\frac{\partial}{\partial y} \approx \frac{\partial}{\partial x},$$

and using (1), we find from (3) the approximation

$$\frac{\partial s}{\partial x} = -\frac{1}{p}\frac{dp}{dx} - \frac{2k}{k-1}\frac{u}{T}\frac{\partial u}{\partial \sigma}, \tag{8}$$

or

$$p\frac{\partial s}{\partial x} = -\frac{dp}{dx} - \frac{2k}{k-1}\left(\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y}\right).$$

From the last equation, using the usual methods of boundary layer theory, we obtain the integral relation

$$p\frac{d}{dx}\int_0^1 s dy = -\frac{dp}{dx} - \frac{2k}{k-1}\frac{d}{dx}\int_0^1 \rho u^2 dy.$$

Here it is necessary to use the equation of continuity.

Introducing the mean mass velocity

$$w = \frac{2}{Q}\int_0^1 \rho u^2 dy,$$

where $Q = 2\int_0^1 \rho u dy$, it is easy to obtain

$$\frac{dp}{dx} + \frac{k}{k-1}Q\frac{dw}{dx} + p\frac{d}{dx}\int_0^1 s dy = 0. \tag{9}$$

Assuming in (8) that $\frac{\partial u}{\partial \sigma} \approx \frac{\partial u}{dx}$, we have

$$\frac{dp}{dx} + \frac{2k}{k-1}\frac{pu}{1-u^2}\frac{\partial u}{\partial x} + p\frac{\partial s}{\partial x} = 0. \tag{10}$$

With the given conditions steady flow can continue until the decreasing ds/dx on one of the stream-lines vanishes. In accordance with (6), this means that on this stream-line conditions are such that transition through the velocity of sound is not possible.

It follows from the measured pressure distribution on the wall of the tube [1, 2] that approach to the critical point is accompanied by a sharp increase in the absolute value of the pressure gradient. It is easy to show by the methods of one-dimensional gas dynamics, that, on the stream-line on which a local critical point occurs, $dp/d\sigma = -\infty$ at that time. Taking this into account, and also condition (7), we shall assume that $dp/dx = -\infty$ at the critical moment. In fact, of course, the phenomenon of criticality is considerably more complex. The model proposed differs from the usual one in that the possibility of distortion of the velocity profile is admitted. Here again, as in the one-dimensional model and in that based on boundary layer theory, the influence of transverse pressure gradients is not taken into account. A more rigorous analysis of this very complex problem is difficult.

Our calculations, and those in [4], show that the quantity $\frac{d}{dx}\int_0^1 s dy$ decreases as the critical point is approach-

At the critical moment therefore, the derivative $\partial s/\partial x$ must be finite on all the stream-lines except at the walls*.

Writing (10) for the channel axis ($u = U$), we obtain

$$\frac{dp}{dx} + \frac{2k}{k-1}\,\frac{pU}{1-U^2}\,\frac{dU}{dx} + p\frac{dS}{dx} = 0. \tag{11}$$

On the basis of (9) and (11), we find

$$\frac{U}{w}\,\frac{d}{dx}\left(\frac{w}{U}\right) = \frac{1}{w}\,\frac{dw}{dx} - \frac{1}{U}\,\frac{dU}{dx} = -\frac{k-1}{2k}\left(\frac{2}{Qw} - \frac{1-U^2}{pU^2}\right)\frac{dp}{dx} + B.$$

Since

$$\frac{2}{Qw} = \frac{1}{\int_0^1 \rho u^2 dy} = \frac{1}{p\int_0^1 [u^2/(1-u^2)]\,dy} \geqslant \frac{1-U^2}{pU^2}$$

(the "equals" sign is possible only when $u = U$), we obtain $\frac{d}{dx}\left(\frac{w}{U}\right) > 0$ as the critical point $\left(\frac{dp}{dx} \to -\infty\right)$ is approached. Thus near the critical point the velocity profile must fill out.

Note that the filling out of the velocity profile as the critical point is approached has been observed experimentally [5].

We shall show that the mean velocity at the critical section is not less than the speed of sound.

Because dS/dx is finite, it follows from (11) that $dU/dx \to \infty$ as the critical point is approached.

From (10) and (11), for the critical section we find

$$\frac{u}{1-u^2}\,\frac{\partial u}{\partial U} = \frac{U}{1-U^2} \tag{12}$$

The condition that mass flow is constant gives

$$\int_0^1 \rho u dy = p\int_0^1 \frac{u}{1-u^2}\,dy = \frac{Q}{2} = \text{const.}$$

Therefore

$$-\frac{1}{p}\,\frac{dp}{dU} = \frac{1}{I}\,\frac{dI}{dU},$$

where

$$I = \int_0^1 \frac{u}{1-u^2}\,dy.$$

Taking (12) into account, we obtain

$$-\frac{1}{p}\,\frac{dp}{dU} = \frac{1}{I}\,\frac{U}{1-U^2}\int_0^1 \frac{1+u^2}{u(1-u^2)}\,dy .$$

From (10), for the critical section we have

$$-\frac{1}{p}\,\frac{dp}{dU} = \frac{2k}{k-1}\,\frac{U}{1-U^2} .$$

*It follows from (10) that at the walls $\partial s/\partial x = -(1/p)\,(dp/dx)$.

Comparing the last two equations, for the critical section, after a small transformation, we have

$$\int_0^1 \frac{u}{1-u^2}\, dy \Big/ \int_0^1 \frac{dy}{u(1-u^2)} = \frac{k-1}{k+1} .$$

Therefore at the critical section

$$w^2 - \frac{k-1}{k+1} = \left(\int_0^1 \frac{u^2}{1-u^2}\, dy\right)^2 \left(\int_0^1 \frac{u}{1-u^2}\, dy\right)^{-2} -$$

$$-\left(\int_0^1 \frac{u}{1-u^2}\, dy\right)\left(\int_0^1 \frac{u}{u(1-u^2)}\, dy\right)^{-1} .$$

Let us replace y by the new variable $z = \int_0^y \frac{dy}{1-u^2}$. Then

$$w^2 - \frac{k-1}{k+1} = \left(\int_0^{z_1} u^2 dz\right)^2 \left(\int_0^{z_1} u dz\right)^{-2} - \left(\int_0^{z_1} u dz\right)\left(\int_0^{z_1} [1/u]\, dz\right)^{-1} , \qquad (13)$$

where

$$z_1 = \int_0^1 \frac{dy}{1-u^2} .$$

Using the Bunyakovskii inequality

$$\int_a^b \varphi^2 dz \int_a^b \psi^2 dz \geqslant \left(\int_a^b \varphi\psi\, dz\right)^2$$

we obtain

$$z_1^2 \left(\int_0^{z_1} u^2 dz\right)^2 \geqslant \left(\int_0^{z_1} u dz\right)^4 .$$

Therefore

$$\left(\int_0^{z_1} u^2 dz\right)^2 \int_0^{z_1} \frac{dz}{u} \geqslant \left(\int_0^{z_1} u dz\right)^3 \frac{1}{z_1^2} \int_0^{z_1} u dz \int_0^{z_1} \frac{dz}{u} . \qquad (14)$$

We shall show that

$$\Delta \equiv \frac{1}{z_1^2} \int_0^{z_1} u dz \int_0^{z_1} \frac{dz}{u} \geqslant 1 .$$

If $u \equiv U$, then $\Delta = 1$. If $\int_0^{z_1} \frac{dz}{u}$ diverges, $\Delta > 1$. Let $\int_0^1 \frac{dz}{u}$ converge and $u \not\equiv U$. We choose a function $f = Uz^m$ such that

$$\int_0^{z_1} \frac{dz}{f} = \int_0^{z_1} \frac{dz}{u}$$

Because of the convexity of the velocity profile, we must have $0 \leq m < 1$. Since for $z = z_1$ from the symmetry of the velocity profile $\frac{\partial}{\partial z}\left(\frac{1}{u}\right) = 0$ and $\frac{\partial}{\partial z}\left(\frac{1}{f}\right) < 0$ the curves $1/f$ and $1/u$ intersect at some $z = z_0$, for $1/u > 1/f$ at $(0, z_0)$ and for $1/u < 1/f$ at (z_0, z_1) (Fig. 1).

Therefore, taking into account the increase in fu with increase in z, and denoting $fu|_{z=z_0}$ by $(fu)_0$, we obtain

$$\int_0^{z_0} (f-u)\,dy = \int_0^{z_0} fu\left(\frac{1}{u}-\frac{1}{f}\right)dy < (fu)_0 \int_0^{z_0}\left(\frac{1}{u}-\frac{1}{f}\right)dy =$$

$$=(fu)_0 \int_{z_0}^{z_1}\left(\frac{1}{f}-\frac{1}{u}\right)dy < \int_{z_0}^{z_1} fu\left(\frac{1}{f}-\frac{1}{u}\right)dy = \int_{z_0}^{z_1}(u-f)\,dy$$

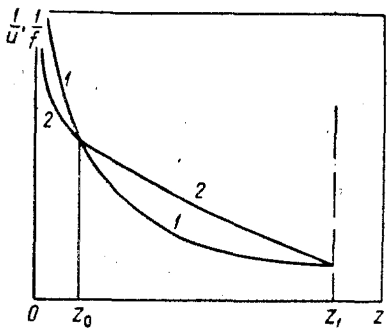

Fig. 1. Curves (schematic) of the functions 1/u (1) and $1/f = Uz^m$ (2).

Consequently,

$$\int_0^{z_1} f dy < \int_0^{z_1} u dy .$$

Therefore

$$\Delta > \frac{1}{z_1^2}\int_0^{z_1} f dy \int_0^{z_1}\frac{dy}{f} = \frac{1}{1-m^2} \geqslant 1 .$$

This inequality, together with (13) and (14), gives

$$w^2 \geqslant \frac{k-1}{k+1} .$$

Thus, the mean velocity at the critical moment must exceed the velocity of sound. It can equal the velocity of sound only when the velocity profile is completely filled out.

Let us estimate the gas velocity at which the velocity profile begins to be fill out, approximating the velocity profile by the relation

$$u = U\,(\eta/\eta_1)^n ,$$

where $\eta = \int_0^y \rho\,dy$ is the Dorodnitsyn variable*.

At the velocities considered, this profile differs little from the usual power law $u = Uy^n$, and, moreover, it means that the relations can be simple.

From the mass flow equation we have

$$\frac{1}{2}Q = \int_0^1 \rho\,u dy = \frac{\eta_1 U}{n+1} ,$$

where the value η_1 is determined from the equation $\int_0^1 \frac{d\eta}{\rho} = 1$,

from which

$$\eta_1\left(1-\frac{U^2}{2n+1}\right) = p ,$$

and consequently

$$p = \frac{n+1}{U}\,\frac{2n+1-U^2}{2n+1}\,\frac{Q}{2} , \qquad (15)$$

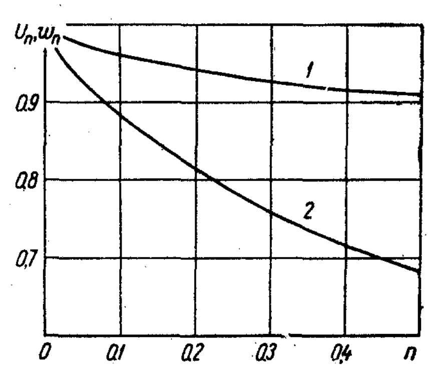

Fig. 2. U_n(1) and w_n(2) as functions of n for $A_1 = 0$. When $U \geq U_n$($w \geq$ $\geq w_n$) the velocity profile fills out.

*Although this profile has a cusp on the axis, it should not lead to appreciable error: the profile is used only in the mass flow equation, where it appears under the integral sign.

Hence, obtaining the value $\frac{1}{p}\frac{dp}{dx}$ and substituting it in (11), we have

$$A_1 \frac{dU}{dx} - A_2 \frac{dn}{dx} = \frac{dS}{dx}, \tag{16}$$

where

$$A_1 = \frac{U^4 - \frac{2k}{k+1}\left(1 + \frac{3k-1}{k} n\right)U^2 + \frac{k-1}{k+1}(2n+1)}{U(1-U^2)(2n+1-U^2)}$$

$$A_2 = \frac{(2n+1)^2 + U^2}{(n+1)(2n+1)(2n+1-U^2)}.$$

The quantity A_2 is always positive. Since $dS/dx \geq 0$, at small velocities, when $A_1 > 0$, increase in U is possible for constant or even increasing n. At a certain velocity $U = U_n$, always less than the velocity of sound (Fig. 2), A_1 vanishes. Then, on the basis of (16), we must have $dn/dx < 0$, i.e., the velocity profile must fill out. Thus, the velocity profile must begin to fill out no later than the time when U reaches the value U_n, at which $A_1(U_n, n) = 0$. As the critical point is approached at the same time as $dU/dx \to \infty$, we see from (16) that $dn/dx \to -\infty$, i.e., the velocity profile very quickly fills out ahead of the critical point, which agrees qualitatively with the results of measurements [5].

NOTATION

2h – channel width, w – flow velocity; T – stagnation temperature; Q – mass flow rate of gas; S – entropy at channel axis; B – some finite quantity.

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